Math 10B - Calculus of Several Variables II - Winter 2011
March 9, 2011
Practice Final

Name: ____

There is no need to use calculators on this exam. All electronic devices should be turned off and put away. The only things you are allowed to have are: a writing utensil(s) (pencil preferred), an eraser, and an exam. All answers should be given as exact, closed form numbers as opposed to decimal approximations (i.e. π as opposed to 3.14159265358979...). Cheating is strictly forbidden. You may leave when you are done. Good luck!

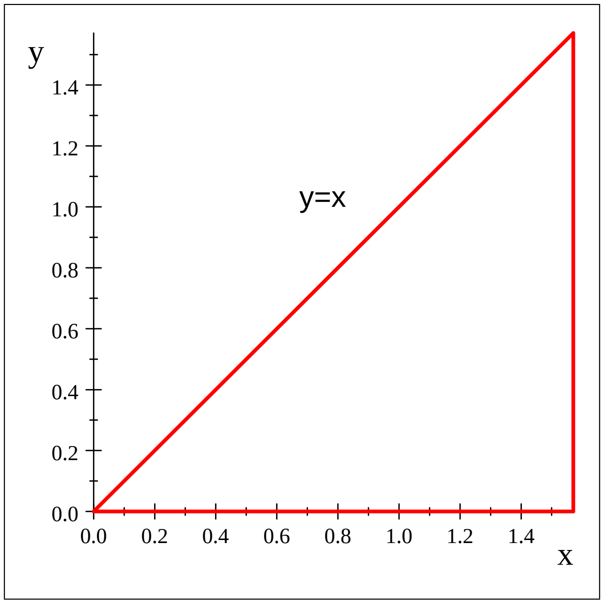
Problem	Score
1	/10
2	/10
3	/20
4	/20
5	/20
6	/20
7	/20
8	/10
9	/20
10	/20
11	
Score	/170

Problem 1 (10 points)**.** *Compute the following integral:*

$$\int_0^{\frac{\pi}{2}} \int_y^{\frac{\pi}{2}} \sin x^2 \; dx dy.$$

Draw the region of integration.

First the region:



Since the integral cannot be integrated in its current form, we must switch the order of integration

$$\int_0^{\frac{\pi}{2}} \int_y^{\frac{\pi}{2}} \sin x^2 dx dy = \int_0^{\frac{\pi}{2}} \int_0^{x} \sin x^2 dy dx = \int_0^{\frac{\pi}{2}} x \sin x^2 dx$$

$$\overset{u=x^2}{=} \frac{1}{2} \int_0^{\frac{\pi^2}{4}} \sin u du = \boxed{-\tfrac{1}{2}\left(\cos \tfrac{\pi^2}{4} - 1\right)}$$

Problem 2 (10 points)**.** *Find the volume of the region bounded by* $z = x^2 + y^2 - 1$ *and* $z = 1 - x^2 - y^2$*.*

Let R be the region above and recall that volume of R is given by

$$V = \iiint_R dV.$$

Notice that this region is easily described in cylidrical coordinates as:

$$0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi, \; r^2 - 1 \leq z \leq 1 - r^2,$$

and so the volume integral is

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \int_{r^2-1}^{1-r^2} r dz dr d\theta = \int_0^{2\pi} \int_0^1 (rz)\big|_{r^2-1}^{1-r^2} dr d\theta = \int_0^{2\pi} \int_0^1 2\left(r - r^3\right) dr d\theta \\ &= \frac{1}{2} \int_0^{2\pi} d\theta = \boxed{\pi} \end{aligned}$$

Problem 3 (20 points)**.**

(a) *(10 points) Compute the Jacobian* $\frac{\partial(x,y)}{\partial(r,\theta)}$ *for changing Cartesian coordinates to polar coordinates.*

(b) *(10 points) Let D be the region bounded by $x^2+y^2=5$ where $x\geq 0$. Compute the integral*

$$\iint_D e^{x^2+y^2}\,dA.$$

(a): Recall the transformation to polar coordinates is given by the map

$$T(r,\theta)=(r\cos\theta, r\sin\theta).$$

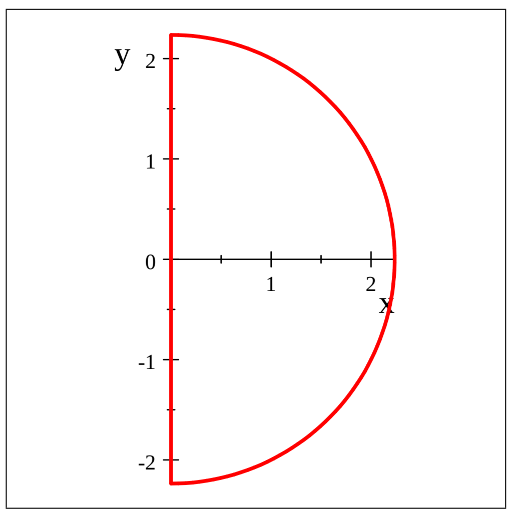
The derivative matrix of this transformation is

$$DT(r,\theta)=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}$$

and hence the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=\boxed{r}.$$

(b): The region of integration, D, looks like



which can be described by $0\leq r\leq\sqrt{5}$ and $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$. Using a substitution to polar coordinates, we have:

$$\begin{aligned}\iint_D e^{x^2+y^2}\,dA &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\sqrt{5}} e^{r^2}r\,dr\,d\theta \overset{u=r^2}{=} \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^5 e^u\,du\,d\theta\\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(e^5-e^0\right)d\theta=\frac{\left(e^5-1\right)}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\\ &= \boxed{\frac{\pi\left(e^5-1\right)}{2}}\end{aligned}$$

Problem 4 (20 points)**.**

(a) *(10 points) Parametrize the circle of radius r.*

(b) *(10 points) Use this parametrization to show that the circumference of the circle of radius r is $2\pi r$. (Hint: Use arclength.)*

(a): The parametrization of the circle of radius r is given by

$$\gamma(t) = (r\cos t, r\sin t)\,,\ 0 \leq t \leq 2\pi.$$

(b): Recall that the arclength of a curve $C : [a, b] \longrightarrow \mathbb{R}^2$ is given by

$$\textit{Arc Length} = \int_C d\mathbf{s} = \int_a^b \|C'(t)\|\, dt$$

and so, using the parametrization above we can find the arclength (i.e. circumference) of the circle of radius r:

First $\gamma'(t) = (-r\sin t, r\cos t)$ and $\|\gamma'(t)\| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r$, so

$$\textit{Arc Length} = \int_0^{2\pi} r dt = 2\pi r.$$

Problem 5 (20 points)**.** *Let C be the boundary of the region bounded by $y = x^2$ and $x = y^2$, oriented counterclockwise.*

(a) *(10 points) Compute the integral*

$$\oint_C \arctan x^3 \, dx + \ln(y^2+1) \, dy.$$

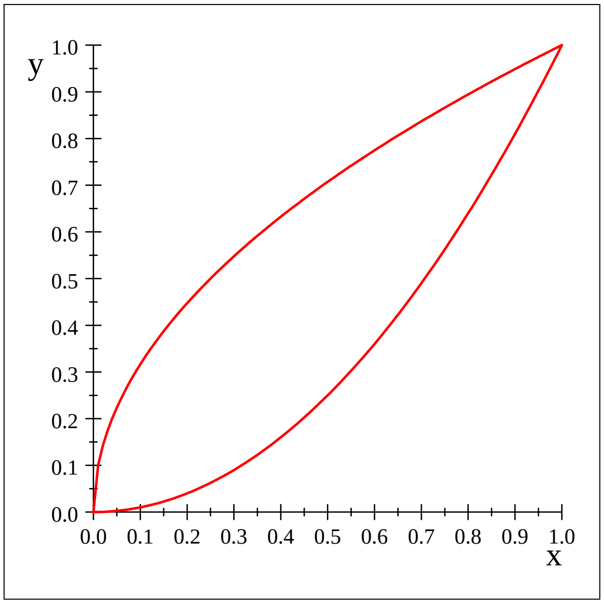
(b) *(10 points) Compute the integral*

$$\oint_C y \, dx - x \, dy.$$

(a): Let R be the region bounded by C. Then, by Green's theorem this integral is

$$\oint_C \arctan x^3 \, dx + \ln(y^2+1) \, dy = \iint_R \left(\frac{\partial}{\partial x}\left[\ln\left(y^2+1\right)\right] - \frac{\partial}{\partial y}\left[\arctan x^3\right] \right) dA = \iint_R (0-0) \, dA = \boxed{0}$$

(b): The region is



So by Green's theorem, the integral is

$$\oint_C y \, dx - x \, dy = \iint_R (-1-1) \, dA = -2 \int_0^1 \int_{\sqrt{y}}^{y^2} dx dy = -2 \int_0^1 \left(y^2 - \sqrt{y}\right) dy = \boxed{\tfrac{2}{3}}$$

Problem 6 (20 points)**.** *Determine whether the following vector fields are conservative. Find a scalar potential function for the ones that are conservative.*

(a) *(10 points)*

$$\vec{\mathbf{F}}(x,y) = (2x\sin y, x^2\cos y).$$

(b) *(10 points)*

$$\vec{\mathbf{G}}(x,y,z) = (y+z, 2z, x+y).$$

(a): Let $M = 2x\sin y$ and $N = x^2\cos y$. Since $\frac{\partial M}{\partial y} = 2x\cos y$ and $\frac{\partial N}{\partial y} = 2x\cos y$, $\boxed{\vec{\mathbf{F}} \text{ is a conservative vector field}}$, so $\vec{\mathbf{F}} = \nabla f$ for some scalar function f. Let's find f :

$$f = \int M dx = \int 2x\sin y dx = x^2\sin y + g(y)$$

and now take a partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = x^2\cos y + g'(y) = N = x^2\cos y$$

so $g'(y) = 0$ and $g(y) = c$, thus

$$\boxed{f(x,y) = x^2\cos y + c.}$$

(b): To check whether or not $\vec{\mathbf{G}}$ is conservative, we need to take its curl

$$\operatorname{curl}\vec{\mathbf{G}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & 2z & x+y \end{vmatrix} = (1-2, -(1-1), 0-1) = (-1, 0, -1) \neq \mathbf{0}$$

thus $\boxed{\vec{\mathbf{G}} \text{ is not conservative}}$.

Problem 7 (20 points)**.** *Let f be a $\mathcal{C}^1$ function on some region $D \subset \mathbb{R}^2$, and consider the surface given by $z = f(x, y)$. Show that the surface area of this surface is given by*

$$S.A. = \iint_D \sqrt{f_x^2 + f_y^2 + 1} dA.$$

Hint: Recall that surface area is given by

$$S.A. = \iint_{\mathbf{X}} d\mathbf{S}$$

where $\mathbf{X}$ is a parametrization of the surface.

First parametrize the surface by $X : D \longrightarrow \mathbb{R}^3$ by

$$X(s,t) = (s, t, f(s,t)).$$

Then

$$T_s = (1, 0, f_s)$$

and

$$T_t = (0, 1, f_t)$$

so the normal vector field is

$$N = T_s \times T_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_s \\ 0 & 1 & f_t \end{vmatrix} = (f_s, -f_t, 1)$$

so $\|N\| = \sqrt{f_s^2 + f_t^2 + 1}$ and hence the surface area is

$$S.A. = \iint_X d\mathbf{S} = \iint_D \|N\| \, dA = \iint_D \sqrt{f_s^2 + f_t^2 + 1} dA.$$

Since s and t were dummy variables, let $s = x$ and $t = y$ to get

$$S.A. = \iint_D \sqrt{f_x^2 + f_y^2 + 1} dA.$$

Problem 8 (10 points)**.** *Let S denote the closed cylinder with bottom given by $z = 0$, top given by $z = 7$, and lateral surface given by $x^2 + y^2 = 49$. Orient S with outward normals. Compute the following integral:*

$$\iint_S (-y\hat{i} + x\hat{j}) \cdot d\mathbf{S}.$$

Let C be the region bounded by the cylinder. Notice that this problem satisfies the hypothesis of Gauss' theorem, so since

$$\text{div}\,(-y\mathbf{i} + x\mathbf{j}) = 0 + 0 = 0$$

by Gauss' theorem

$$\iint_S (-y\hat{\imath} + x\hat{\jmath}) \cdot d\mathbf{S} = \iiint_C 0 dV = \boxed{0}.$$

Problem 9 (20 points)**.** *Let S be the sphere given by $x^2 + y^2 + z^2 = 1$ with outward pointing normals.*

(a) *Let $\mathbf{F}(x, y, z) = (2xyz + 5z, e^x \cos yz, x^2 y)$. Compute*

$$\iint_S \mathit{curl}\, \mathbf{F} \cdot d\mathbf{S}.$$

(b) *Let $\mathbf{G}(x, y, z) = (x, y, z)$. Compute*

$$\iint_S \mathbf{G} \cdot d\mathbf{S}.$$

Hint: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

(a): Notice that is problem satisfies the hypothesis of Stokes' theorem. Also, recall that the boundary of a sphere is empty (i.e. the sphere does not have a boundary), so by Stokes' theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \boxed{\mathbf{0}}.$$

(b): Let B be the ball bounded by the sphere above. Then B is the ball of radius 1, so its volume is $\frac{4}{3}\pi$. This problem satisfies Gauss' theorem, so since $\operatorname{div} \mathbf{G} = 1 + 1 + 1 = 3$, by Gauss' theorem

$$\begin{aligned} \iint_S \mathbf{G} \cdot d\mathbf{S} &= \iiint_B \operatorname{div} \mathbf{G} dV = 3 \iiint_B dV \\ &= 3\,(\text{volume of ball of radius } 1) \\ &= 3\left(\frac{4}{3}\pi\right) = \boxed{4\pi} \end{aligned}$$

Problem 10 (20 points)**.** *Verify that Stokes' theorem implies Green's theorem. Hint: Use the vector field* $\mathbf{F}(x, y, z) = (M(x, y), N(x, y), 0)$*.*

Recall the statements of Stokes' and Green's theorems

Theorem (Stokes' Theorem)**.** *Let* S *be a bounded, piecewise smooth, oriented surface in* $\mathbb{R}^3$*. Suppose that* ∂S *consists of finitely many piecewise* $\mathcal{C}^1$*, simple, closed curves each of which is oriented consistently with* S*. Let* $\mathbf{F}$ *be a vector field of class* $\mathcal{C}^1$ *whose domain includes* S*. Then*

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Theorem (Green's Theorem)**.** *Let* D *be a closed, bounded region in* $\mathbb{R}^2$ *whose boundary* $C = \partial D$ *consists of finitely many simple, closed curves. Orient the curves of* C *so that* D *is on the left as one traverses* C*. Let* $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ *be a vector field of class* $\mathcal{C}^1$ *throughout* D*. Then*

$$\oint_C M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$. Let us further assume that S lies in the xy-plane. Let $C = \partial S$ and assume in addition to the above assumptions that C consists of finitely many simple, closed curves oriented so that S is on the left as you traverse C. Since $z = 0$, let $\mathbf{F}(x, y, z) = (M(x, y), N(x, y), 0)$. Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

Now by Stokes' theorem

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{s} = \oint_C M dx + N dy.$$

Thus putting the two equations together we have

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_C M dx + N dy$$

and we are done.